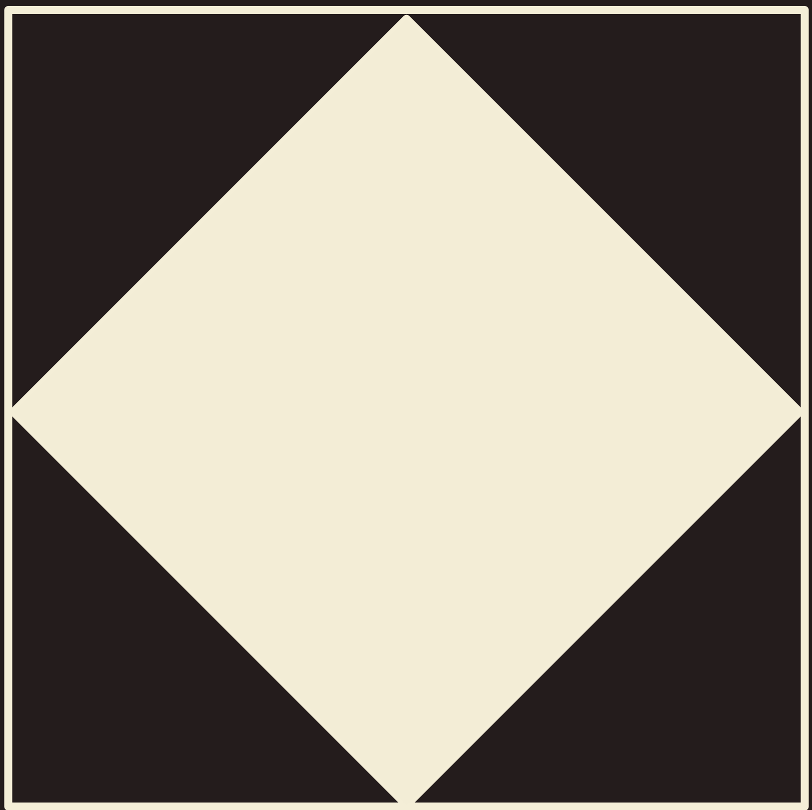
TOPICS IN MATHEMATICS
TRANSLATED FROM THE RUSSIAN

HOW TO CONSTRUCT GRAPHS

G. E. SHILOV

SIMPLEST MAXIMA AND MINIMA PROBLEMS

I. P. NATANSON



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These American editions have been prepared by the Survey of Recent East European Mathematical Literature at the University of Chicago under a grant from the National Science Foundation.

Titles of the series *Topics in Mathematics* are listed on the inside back cover.

T O P I C S   I N   M A T H E M A T I C S

How to Construct Graphs

G. E. Shilov

Translated and adapted from the first Russian edition (1959) by

JEROME KRISTIAN *and* DANIEL A. LEVINE

Simplest Maxima and Minima Problems

I. P. Natanson

Translated and adapted from the third Russian edition (1960) by

C. CLARK KISSINGER *and* ROBERT B. BROWN

SURVEY OF

RECENT EAST EUROPEAN MATHEMATICAL LITERATURE

A project conducted by

ALFRED L. PUTNAM *and* IZAAK WIRSZUP

Department of Mathematics,
The University of Chicago, under a
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PREFACE TO THE AMERICAN EDITION

THE FIRST PART of this booklet, *How to Construct Graphs* by G. E. Shilov, presents simple methods of plotting graphs, first "by points" and then "by operations." The latter method offers a means of constructing graphs of complicated functions by considering the function as a succession of operations performed on an initial quantity.

The second part, *Simplest Maxima and Minima Problems* by I. P. Natanson, shows how to solve certain maxima and minima problems by algebraic methods. This material is excellent preparation for calculus, in which such problems are treated more generally. (In order to relate this part to the preceding one, several paragraphs and Fig. A and Fig. B, not present in the Russian edition, have been added.)

This booklet can be read by anyone who has studied intermediate algebra.

CONTENTS

How to Construct Graphs

G. E. SHILOV

1. Graphs "by Points"

1. INTRODUCTION

There are few areas of science or everyday life where graphs cannot be of some use. For example, we have all seen graphs showing production statistics or records of natural phenomena, such as daily or yearly variation in temperature, atmospheric pressure, etc. Given tables of data, it is not hard to construct such graphs.

In this booklet, we shall discuss a different kind of graph, however. We shall discuss graphs which are constructed from mathematical equations. Such graphs are of wide interest. For example, a scientist analyzing the theoretical operation of a hypothetical chemical process may derive an equation which gives the quantity of end product produced as a function of time. The graph of this equation will show the results of the proposed process visually. Upon looking at the graph, the scientist may introduce some change in the plan of the experiment so as to give better results.

2. COORDINATE SYSTEM

Let us draw two mutually perpendicular straight lines in a plane, one of them horizontal and the other vertical, and call their point of intersection O. We shall call the horizontal line the *x-axis* and the vertical line the *y-axis*. The point O divides each axis into a *positive* semiaxis and a *negative* semiaxis: the right semiaxis of the x-axis and the upper semiaxis of the y-axis are considered positive; the left semiaxis of the x-axis and the lower semiaxis of the y-axis are considered negative. We indicate the positive semiaxes by arrows (see Fig. 1).

The location of any point M in the plane can now be specified by a pair of numbers. To do this, we construct the perpendiculars MA and MB from M to the axes. These cut off segments OA and OB on the axes (see Fig. 1). The length of the seg-

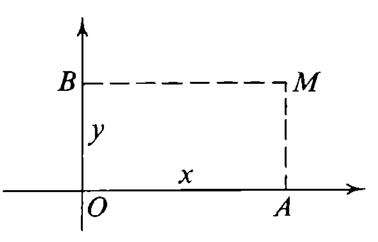


Fig. 1

ment OA, taken with a $+$ sign if A lies on the positive semiaxis or with a $-$ sign if A lies on the negative semiaxis, is called the *abscissa* of the point M and is denoted by x. Similarly, the length of the segment OB, taken with the sign determined by the same rule, is called the *ordinate* of the point M and is denoted by y. The two numbers x and y are called the *coordinates* of the point M.

Each point in the plane has definite coordinates. Points on the x-axis have ordinates equal to zero and points on the y-axis have abscissas equal to zero. The point of intersection O, called the *origin*, has both coordinates equal to zero.

On the other hand, given any two (positive or negative) numbers x and y, we can always find a point M with abscissa x and ordinate y. To do this, it is necessary to construct (in the proper direction) on the x-axis the line segment OA whose length is x, and from the point A construct (in the proper direction) the perpendicular AM of length y. This locates the point M.

3. GRAPH OF AN EQUATION

Suppose that we are given an equation and are asked to construct its graph. The equation must tell us what operations are to be performed on the *independent variable* x in order to produce the value of the *dependent variable* y. For example, the equation

$$y = \frac{1}{1 + x^2}$$

says that the value of the quantity y is to be obtained from the value of x by squaring x, adding 1 to it, and dividing 1 by the result. If x is given some numerical value x_0, y will take on a numerical value y_0 which is defined by the equation. The two numbers x_0 and y_0 define a point M_0 in the plane. If x is given some other value x_1, y will take on a value y_1, and the pair of numbers (x_1, y_1) will define a second point M_1 in the plane. The locus of *all* such points, whose ordinates are derived from their abscissas by the equation, is called the *graph* of the equation.

In general, the number of points on a graph is infinite, and so we cannot possibly construct every one. Nevertheless, we can get around this difficulty. In most cases, some small number of points suffices to show the general shape of a graph. To construct a graph "by points," we construct a number of points on the graph and join them by a smooth curve.

As an example, we shall construct the graph of the equation

$$y = \frac{1}{1 + x^2}. \tag{1}$$

From the equation we can draw up the following table:

x	0	1	2	3	-1	-2	-3
y	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

In the first row we have written the values

$$x = 0,\ 1,\ 2,\ 3,\ -1,\ -2,\ -3.$$

We usually choose whole numbers for x in order to simplify the calculations. In the second row we have written the corresponding values of y, derived from equation (1). Figure 2 shows the cor-

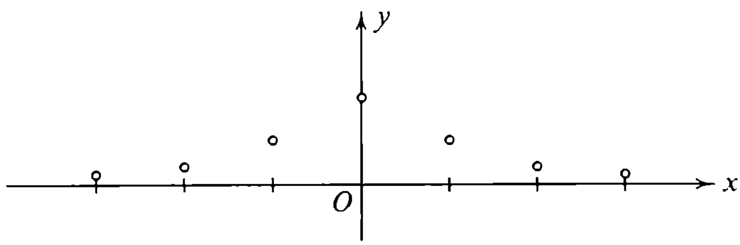


Fig. 2

responding points in the plane. Joining these points by a smooth curve, we obtain the graph shown in Figure 3.

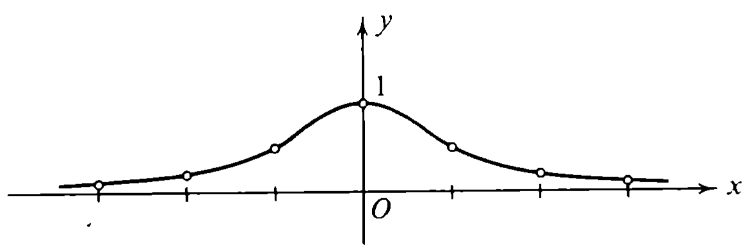


Fig. 3

The method of constructing graphs "by points" is extremely simple and requires no "science." Nevertheless, precisely because it is so simple, following it blindly can lead to large errors.

Let us draw the graph of

$$y = \frac{1}{(3x^2 - 1)^2} \tag{2}$$

"by points." The table of values of x and y corresponding to this equation is:

x	0	1	2	3	-1	-2	-3
y	1	$\frac{1}{4}$	$\frac{1}{121}$	$\frac{1}{676}$	$\frac{1}{4}$	$\frac{1}{121}$	$\frac{1}{676}$

The corresponding points in the plane are shown in Figure 4. This

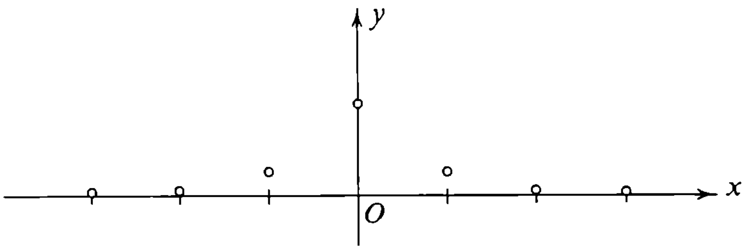


Fig. 4

drawing strongly resembles the one in Figure 2. Joining the points by a smooth curve, we obtain the graph of Figure 5. It would seem

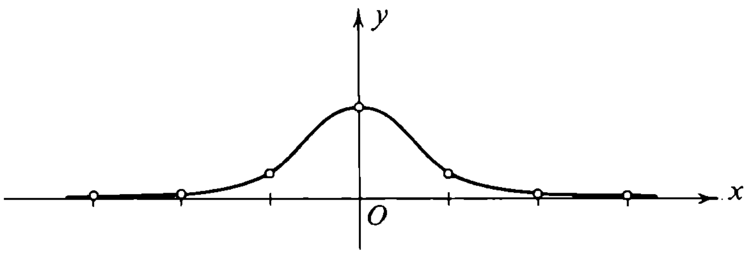


Fig. 5

that we could put down our pencils and relax, since we have now mastered the art of drawing graphs. But first, as a check, let us calculate y for some intermediate value of x, say $x = 0.5$. This gives the unexpected result $y = 16$. This hardly corresponds to our drawing. Nor can we guarantee that for some other intermediate value of x (there are an infinite number of them) the calculated value of y will not be even further from the drawing. There must be some fundamental insufficiency inherent in the method of constructing graphs "by points." We shall clarify this in the next chapter.

2. Graphs "by Operations"

We shall now consider another method of constructing graphs which is more promising in the sense that it is safe from surprises such as the one we have just encountered. In this method—we shall call it the "method of operations"—it is necessary to perform each of the operations appearing in the given formula, addition, subtraction, multiplication, division, etc., directly on the graph.

4. GRAPHS OF FIRST-DEGREE EQUATIONS

First, let us construct the graph corresponding to the equation

$$y = x.$$

This equation says that all points on its graph have equal abscissas and ordinates. The locus of such points is the bisector of the angle between the positive semiaxes and of the angle between the negative semiaxes (Fig. 6).

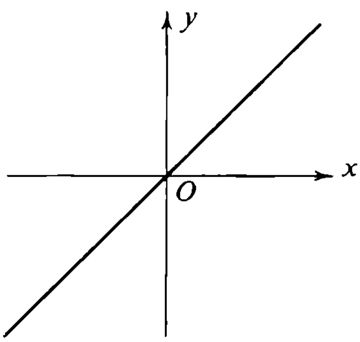


Fig. 6

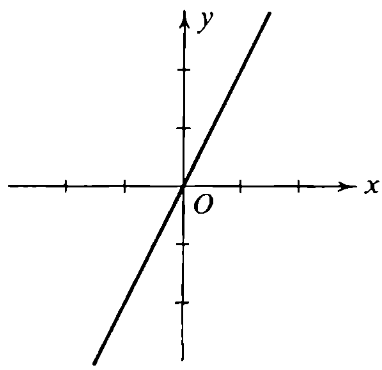


Fig. 7

The graph of the equation

$$y = kx,$$

where k is an arbitrary but fixed coefficient, is obtained from the preceding graph by multiplying each of its ordinates by k. For example, suppose $k = 2$: the graph is obtained by doubling the ordinate of each point of the graph of Figure 6, and the result is a more steeply rising straight line (Fig. 7). For each unit to the right along

the x-axis the new line rises two units in the positive direction of y, instead of one. This makes it easy to construct the graph on graph paper. In general, the graph of the equation $y = kx$ is always a straight line; if $k > 0$, for each unit to the right, it will rise k units in the positive direction of y; if $k < 0$, it will drop rather than rise.

Next consider the somewhat more complicated equation

$$y = kx + b. \tag{4}$$

To construct its graph, we must add the fixed number b to each ordinate of the already-known graph $y = kx$. This means that the entire line $y = kx$ will be shifted, as a whole, b units upward in the plane if $b > 0$. If $b < 0$, of course, the line $y = kx$ will be lowered rather than raised. The result is a straight line which is parallel to the line $y = kx$, but which, instead of passing through the origin, intersects the y-axis at distance b from the origin (Fig. 8).

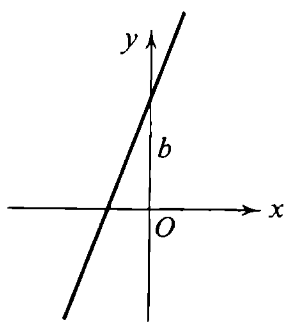


Fig. 8

Thus, *the graph of any equation of the first degree is a straight line,* which can be drawn using the rules we have just discussed.

5. GRAPHS OF SECOND-DEGREE EQUATIONS

Consider the equation

$$y = x^2. \tag{5}$$

This can be written in the form

$$y = y_1^2,$$

where

$$y_1 = x.$$

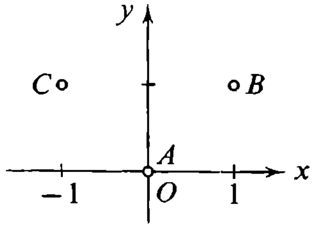


Fig. 9

In other words, the graph can be obtained by squaring the ordinate of each point of the straight line $y = x$.

Let us see what happens in this process. Since $0^2 = 0$, $1^2 = 1$, $(-1)^2 = 1$, we have three reference points, A, B, and C of the graph (Fig. 9).

If $x > 1$, $x^2 > x$; therefore, to the right of the point B, the graph rises above the bisector of the angle between the positive semiaxes (that is, above the line $y = x$), as in Figure 10.

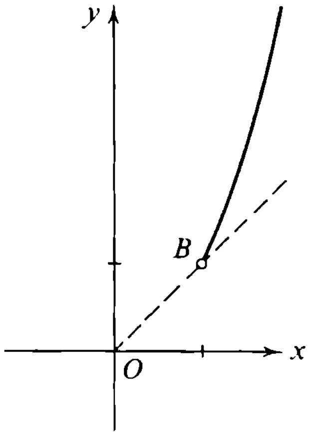


Fig. 10

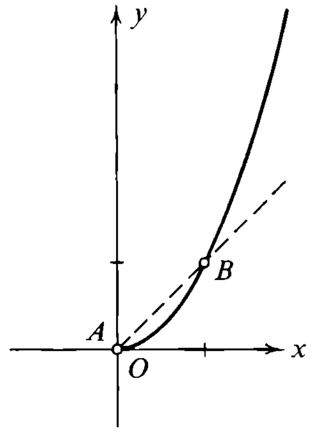


Fig. 11

If $0 < x < 1$, then $0 < x^2 < x$; therefore, between points A and B the graph lies below the bisector, as in Figure 11. Furthermore, the graph approaches the point A within the angle formed by the line $y = kx$ (where k is as small as we please) and the x-axis. This follows since $x^2 < kx$ for all $x < k$. This fact shows that the graph is *tangent* to the x-axis at the point O (Fig. 11).

Let us now consider the portion of the graph which lies to the left of the y-axis. We know that the numbers $-a$ and $+a$ both have the same square $(+a^2)$. Thus, the ordinate at $x = -a$ will be the same as the ordinate at $x = +a$. Geometrically this means that the portion of the graph on the left side of the y-axis is a mirror image of the portion of the graph on the right side of the y-axis. The curve which we obtain from the operation of squaring each ordinate of the line $y = x$ is called a *parabola* (Fig. 12).

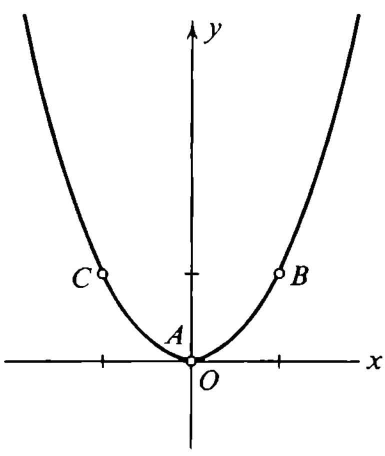


Fig. 12

Starting with the curve for

$$y = x^2,$$

which we shall call the *standard* parab-

ola, we can construct the graph of the more complicated equation

$$y = ax^2 \tag{6}$$

and of the still more complicated equation

$$y = ax^2 + b. \tag{7}$$

The graph of equation (6) is obtained by multiplying each ordinate of the standard parabola by a. If $a > 1$, the curve is similar to Fig. 12, but rises more steeply (Fig. 13); if $0 < a < 1$, the curve is flatter (Fig. 14). If $a < 0$, it is inverted (Fig. 15).

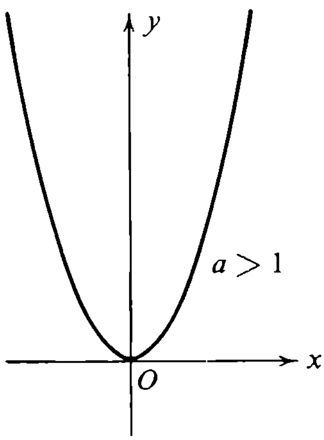


Fig. 13

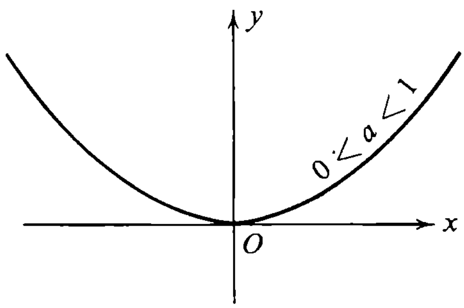


Fig. 14

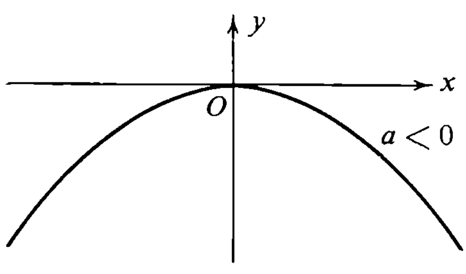


Fig. 15

The graph of equation (7) is obtained from the graph of equation (6) by shifting it upward a distance b if $b > 0$ (Fig. 16) or

downward a distance b if $b < 0$ (Fig. 17). All these curves are also parabolas.

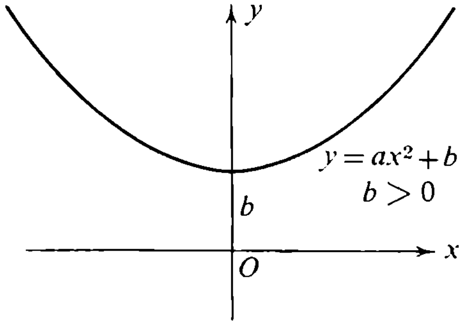


Fig. 16

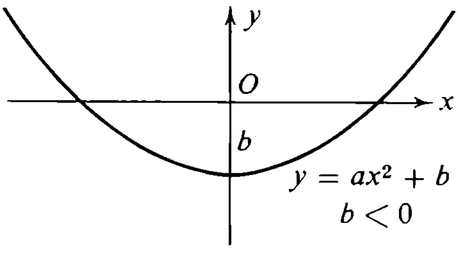


Fig. 17

6. GRAPHS BY MULTIPLICATION

We next consider a more complicated example—the construction of graphs by the method of multiplication. Suppose we must draw the graph of the equation

$$y = x(x - 1)(x - 2)(x - 3). \tag{8}$$

Here y is the product of four factors. We begin by drawing the graph of each factor individually. Each one is a straight line, parallel to the bisector of the angle between the positive semiaxes and intersecting the y-axis at 0, -1, -2, and -3, respectively (Fig. 18).

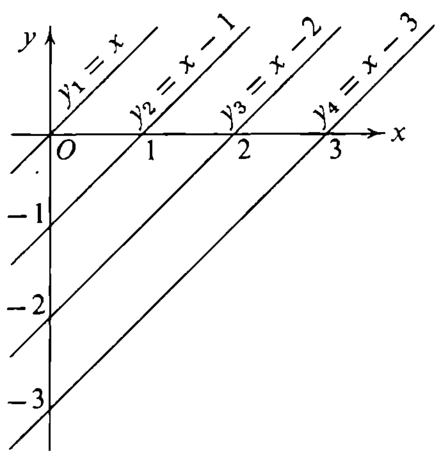


Fig. 18

At $x = 0$, 1, 2, and 3 the graph of (8) will have the ordinate zero, since a product is zero when any of its factors is zero. Elsewhere, the product will be different from zero and will have a sign which can easily be found from the signs of the factors.

Since all the factors are positive to the right of the point 3, the product must also be positive in this region.

Between the points 2 and 3 one factor is negative and the others are positive; so the product is negative.

Between the points 1 and 2 there are two negative factors and two positive factors; therefore, the product is positive.

Between the points 0 and 1 there are three negative factors and one positive factor; here the product is negative.

To the left of point 0 all four factors are negative; hence, the product is positive.

Thus, we obtain the pattern of signs shown in Fig. 19.

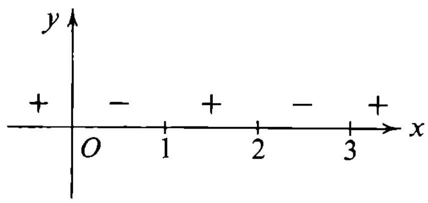


Fig. 19

To the right of the point 3, each factor grows larger with increasing x, and so the product grows larger—quite rapidly. To the left of the origin O, each factor grows larger in the negative direction; so the product (which is positive) will again increase rapidly. It is now easy to sketch the general form of the graph (Fig. 20).

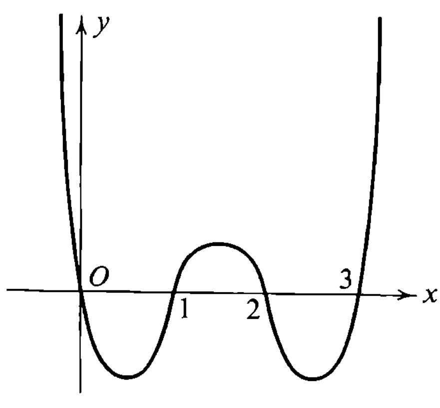


Fig. 20

7. GRAPHS BY DIVISION

So far we have used the operations of addition and multiplication. Let us now add division. We shall construct the graph of

$$y = \frac{1}{1 + x^2}. \tag{9}$$

We begin by constructing separately the graphs of the numerator and denominator. The graph of the numerator,

$$y_1 = 1,$$

is a straight line parallel to the x-axis and one unit above it. The graph of the denominator,

$$y_2 = x^2 + 1,$$

is a standard parabola moved upward one unit. Both of these graphs are shown in Figure 21.

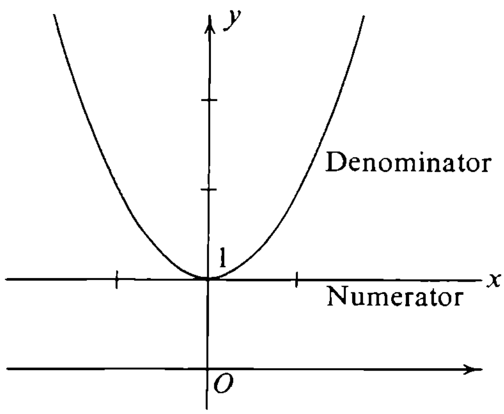


Fig. 21

We next divide each ordinate of the numerator by the corresponding ordinate (that is, belonging to the same x) of the denominator. For $x = 0$, $y_1 = y_2 = 1$, and so $y = 1$. For $x \neq 0$, the numerator is smaller than the denominator and their quotient is less than one. Since both numerator and denominator are always positive, their quotient is also always positive, and must therefore lie entirely within the strip of the plane bounded by the line $y = 1$ and the x-axis. As x becomes arbitrarily large, so does the denominator. Since the numerator remains constant, the quotient must

approach zero. We thus obtain the graph of the quotient (Fig. 22). It is the same graph that we obtained "by points" earlier (Fig. 3).

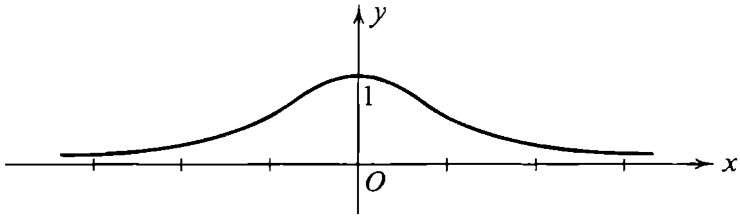


Fig. 22

In graphical division the values of x for which the denominator vanishes (becomes equal to zero) must receive special attention. If the numerator does not also vanish at the same point, the quotient becomes infinite there.

As an example, let us draw the graph of

$$y = \frac{1}{x}. \tag{10}$$

We already know the graphs of the numerator and denominator (Fig. 23). At $x = 1, y_1 = y_2$; hence, $y = 1$. For $x > 1$, the numer-

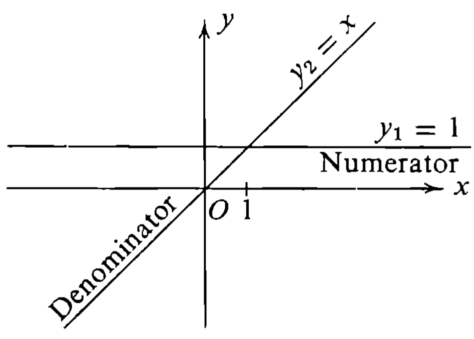


Fig. 23

ator is smaller than the denominator and the quotient is therefore less than one, as in the last example. As x becomes arbitrarily large, the quotient approaches zero. We may now draw the section of the graph corresponding to values of $x > 1$ (Fig. 24).

Now consider values of x between zero and one. As x decreases from one and approaches zero, the denominator approaches zero but the numerator remains equal to one. Therefore, the quotient

becomes large without bound, and this section of the graph rises steeply (Fig. 25). For $x < 0$, both the denominator and the entire

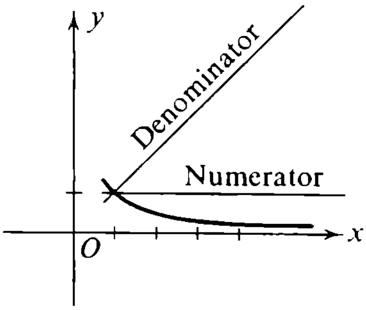


Fig. 24

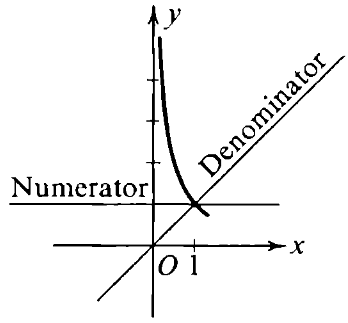


Fig. 25

fraction are negative. The general shape of the graph is shown in Figure 26.

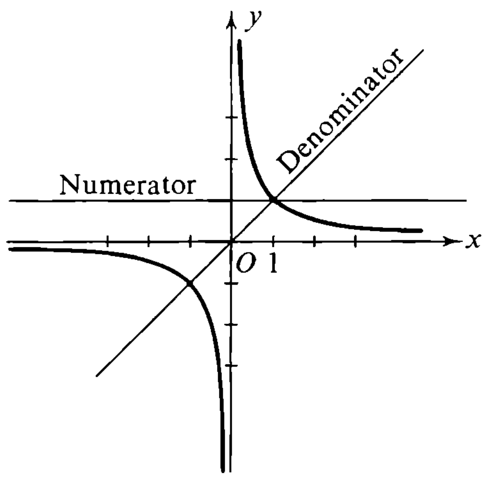


Fig. 26

We can now properly construct the graph of the equation which gave us trouble earlier:

$$y = \frac{1}{(3x^2 - 1)^2}. \tag{11}$$

We begin by constructing the graph of the denominator. First, $y_1 = 3x^2$ is a "tripled" standard parabola (Fig. 27). Subtracting 1 has the effect of lowering the entire figure one unit (Fig. 28). The

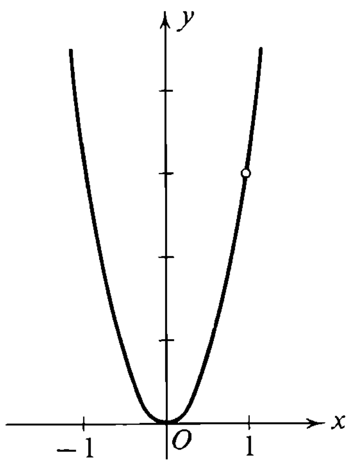


Fig. 27

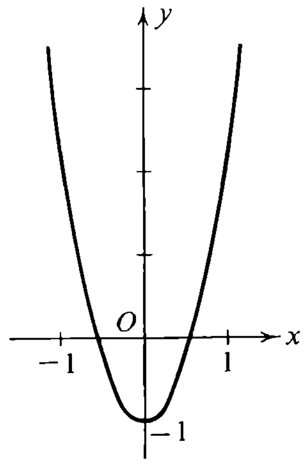


Fig. 28

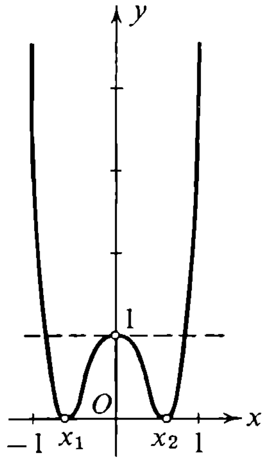


Fig. 29

graph of the denominator intersects the x-axis in two points, which are easily found by setting $3x^2 - 1$ equal to zero:

$$x_{1,2} = \pm\sqrt{\frac{1}{3}} = \pm 0.577\ldots$$

We next square the graph of $3x^2 - 1$. At the points x_1 and x_2 the ordinates remain zero, while all the other ordinates are positive, so that the graph lies entirely on or above the x-axis. At $x = 0$, the ordinate is $(-1)^2 = 1$, and this is the largest value of the ordinates between x_1 and x_2. Outside this strip the curve rises very steeply (Fig. 29).

We have constructed the graph of the denominator. Also in Figure 29 we have shown the graph of the numerator, $y_4 = 1$, in dashes. We must now divide the numerator by the denominator. Since both have the same sign (positive) everywhere, the fraction is positive everywhere, and so the entire graph lies above the x-axis. At $x = 0$, the numerator and denominator are equal; so their ratio is equal to 1. Moving to the right along the x-axis from the origin O, the numerator remains equal to 1, and the denominator steadily decreases. This means that their quotient steadily *increases* from the value 1. At the point $x_2 = 0.577\ldots$, the denominator is zero. Therefore, the quotient becomes infinite at this point (Fig. 30). Beyond the point x_2, the denominator quickly goes from 0 to 1, and continues to increase without bound. The quotient, on the other hand, drops from infinity to 1, intersects the line $y = 1$, and then gets closer and closer to 0 as x increases (Fig. 31).

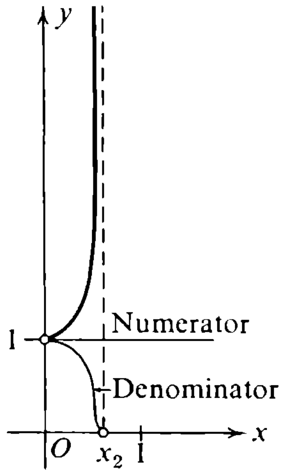


Fig. 30

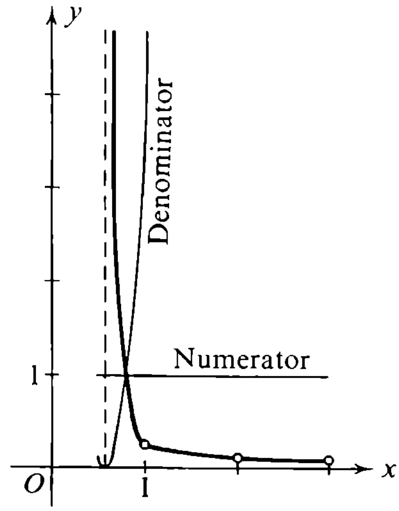


Fig. 31

The graph to the left of the y-axis is the mirror image of that to the right (Fig. 32).

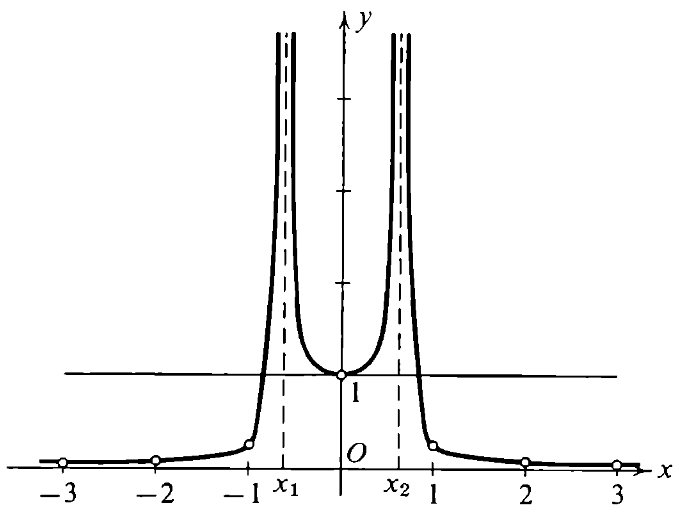


Fig. 32

We have shown the points on the graph of Figure 32 which correspond to the whole numbers $x = 0, 1, 2, 3, -1, -2, -3$. These are the same points which we plotted when we tried to construct the graph "by points" in Figure 5. The actual shape of the graph (Fig. 32) differs markedly from the curve in Figure 5. Instead of dropping smoothly from 1 (at $x = 0$) to $\frac{1}{4}$ (at $x = 1$), it rises in-

definitely. Note also that Figure 32 clearly shows the point with coordinates $y = 16$, $x = \frac{1}{2}$, while this point is nowhere near the curve shown in Figure 5.

8. SUMMARY

We shall conclude by summarizing the general rules for constructing graphs "by operations":

(*a*) Each of the operations indicated in the equation is performed graphically, going from the simplest to the more complicated.

(*b*) When "multiplying graphs," pay special attention to the points at which the factors (even if there is only one of them) become zero; between these points, remember the rule for the sign of a product.

(*c*) When "dividing graphs," pay special attention to the points at which the denominator is zero. Unless the numerator also is zero at such a point, the quotient will become infinite (in the positive or negative direction, depending upon the signs of numerator and denominator).

(*d*) Notice the behavior of the curve as x increases without bound to the right (to $+\infty$) and to the left (to $-\infty$).

We have discussed only the simpler operations which can be performed graphically. More precisely, we began with the very simple equation $y = x$ and applied the four arithmetic operations (addition, subtraction, multiplication, and division). To these we might easily have added the algebraic operation of extracting roots.

More complicated operations—trigonometric and logarithmic—can also be performed graphically. We would only need to have at hand the graphs of the two basic equations $y = \sin x$ (Fig. 33)

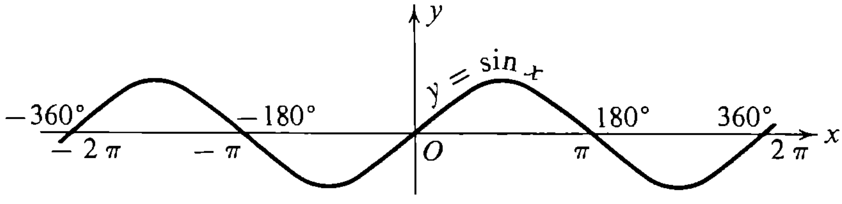


Fig. 33

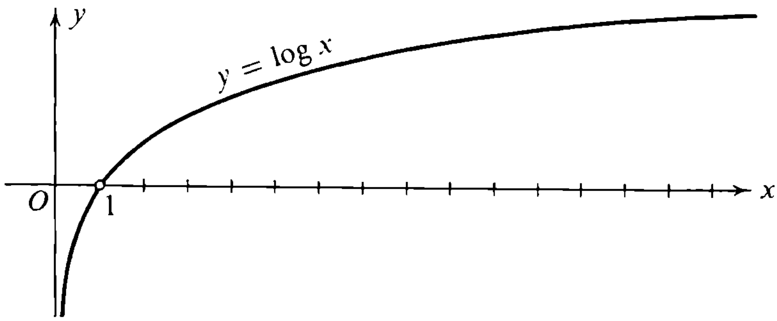


Fig. 34

and $y = \log x$ (Fig. 34). Using the methods discussed above, we could then construct the graph of any equation involving arithmetic, algebraic, trigonometric, or logarithmic operations.

While it is useful to learn how to draw a variety of graphs, the methods we have shown above cannot be used to answer many questions we might naturally ask about particular graphs. For example, we have seen that in some graphs, the curve rises to a particular value y_0 and then decreases again. That is, at some point x_0 it reaches a maximum value, y_0. Using our limited arsenal of methods, we cannot always be certain of finding the exact value of such x_0's. Also, we might be interested in finding the angles at which a curve intersects the x- and y-axes, or whether the curve opens upwards or downwards. To answer these and other important questions, we need the command of some more powerful mathematical techniques. The methods which we need in order to investigate these properties of graphs are developed in the field of mathematics known as *differential calculus.*

Exercises and Solutions

Draw the graphs of the following equations:

1. $y = x^2 + x + 1$
2. $y = x(x^2 - 1)$
3. $y = x^2(x - 1)$
4. $y = x(x - 1)^2$
5. $y = \dfrac{x}{x - 1}$ *Hint.* Separate the integral part: $\dfrac{x}{x-1} = 1 + \dfrac{1}{x - 1}$
6. $y = \dfrac{x^2}{x - 1}$ *Hint.* Separate the integral part.
7. $y = \dfrac{x^3}{x - 1}$ *Hint.* Separate the integral part.
8. $y = \pm \sqrt{x}$

Hint. The square root of a negative number does not exist in the real domain.

9. $y = \pm\sqrt{1 - x^2}$. Prove that this curve is a circle.

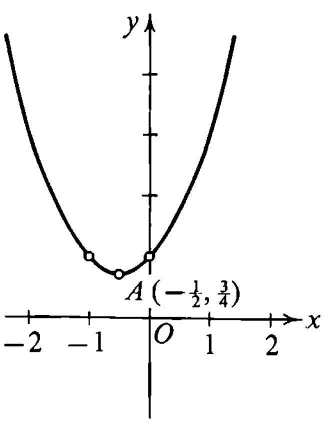
Hint. Use the definition of a circle and use the Pythagorean theorem.

10. $y = \pm \sqrt{1 + x^2}$. Prove that the upper branch of the curve comes arbitrarily close to the bisector of the angle between the positive semiaxes as $x \to \infty$.

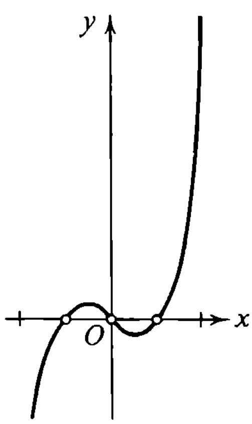
Hint. $\sqrt{x^2 + 1} - x = \dfrac{1}{\sqrt{x^2 + 1} + x}$

11. $y = \pm x\sqrt{x(1 - x)}$
12. $y = \pm x^2\sqrt{1 - x}$
13. $y = \dfrac{1 - x^2}{2 \pm \sqrt{1 - x^2}}$
14. $y = x^{\frac{2}{3}}(1 - x)^{\frac{2}{3}}$

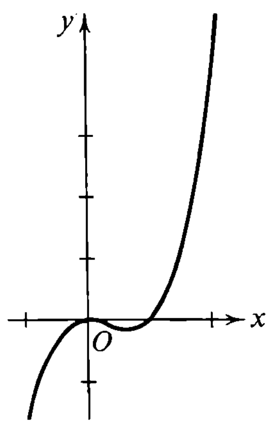
SOLUTIONS OF EXERCISES



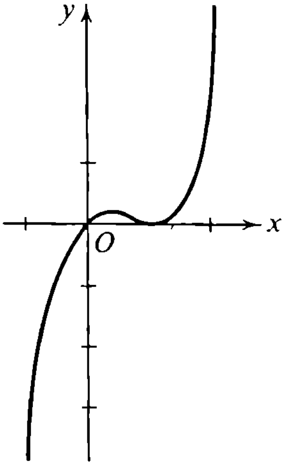
Exercise 1



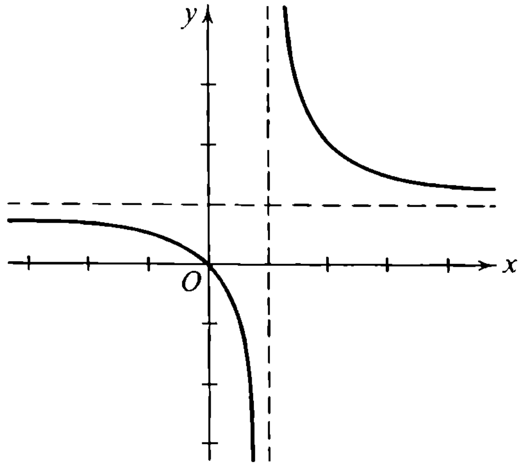
Exercise 2



Exercise 3



Exercise 4



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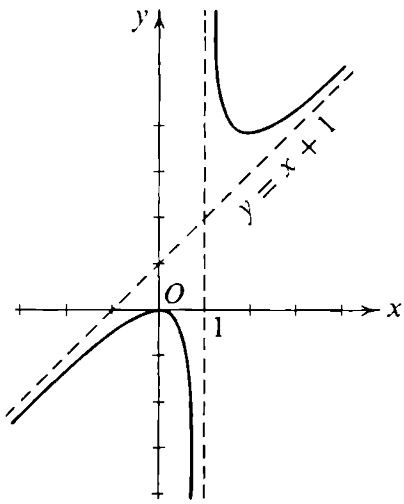
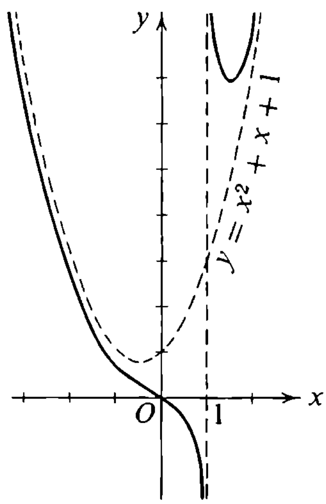
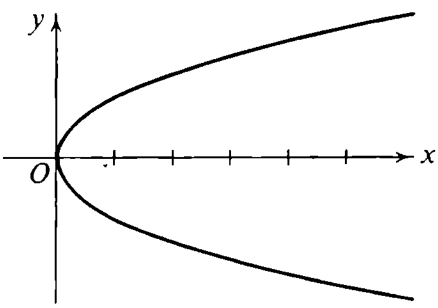


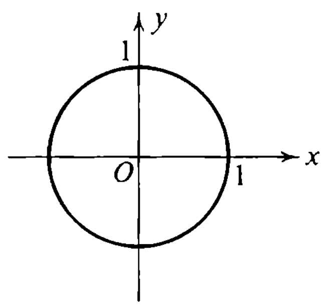
Exercise 6



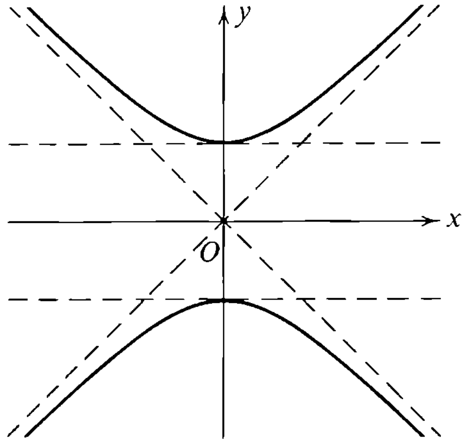
Exercise 7



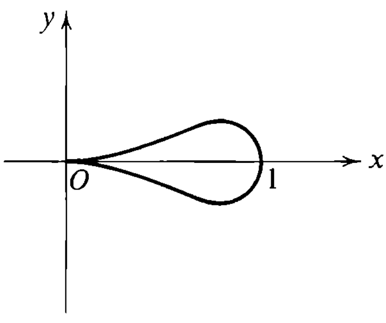
Exercise 8



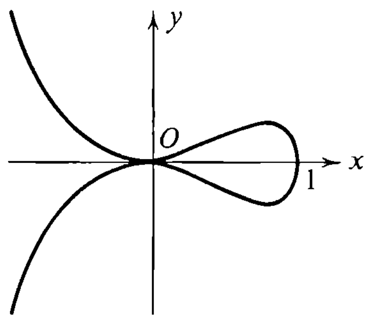
Exercise 9



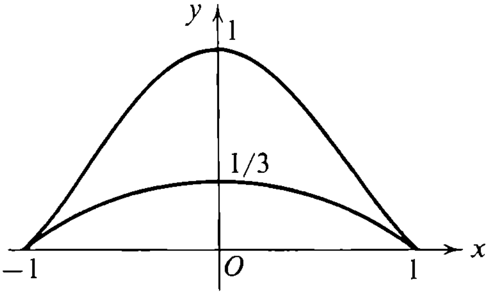
Exercise 10



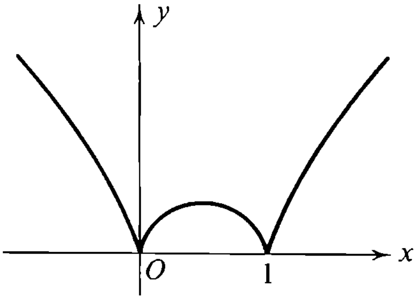
Exercise 11



Exercise 12



Exercise 13



Exercise 14

Simplest Maxima and Minima Problems

I. P. NATANSON

Introduction

In engineering and in science, in industry and in our daily life, we encounter certain mathematical problems called "maxima and minima problems." Here are a few examples:

1. A rectangular beam is to be sawed from a circular tree trunk so that there is as little waste as possible.
2. A fence 200 meters long can be constructed from available material. A rectangular piece of land of the greatest possible area is to be enclosed within this fence, using a factory wall as one side of the yard.
3. A picture hangs on the wall above eye level. At what distance from the wall must an observer stand in order that the angle subtended by the picture will be the greatest?
4. How high above the center of a round table must a lamp be hung in order to get the strongest illumination at the table edge?

All of these problems have something in common in spite of their differences. Each is a question of how to achieve the optimal effect by choosing one of a variety of possibilities. The importance of being able to solve such problems need not be emphasized. Mathematics has created some very useful and general methods for the solution of such problems. They are studied in differential calculus.

However, it is possible in many cases to solve such problems using only the simple methods of elementary algebra, without bringing in the complicated apparatus of differential calculus. Several such methods for the solution of maxima and minima problems without recourse to higher mathematics will be treated in this booklet.[1] Of course, such methods are applicable only in individual cases. Nevertheless, familiarity with them is often of use even to those who possess a knowledge of differential calculus.

[1] In particular, the four problems cited above will be solved; see Problems 6, 3, 11, 17.

1. The Fundamental Theorem on Quadratic Trinomials

1. PARABOLAS; MINIMUM VALUES

Consider two variables x and y which are connected by the equation

$$y = 2x^2 + 7. \qquad (*)$$

The graph of this equation, plotted by the methods described by G. E. Shilov in the preceding part, is a *parabola* and is shown in Figure A. Notice that only a portion of the entire graph is shown. It would take an infinitely large sheet of paper to show it all.

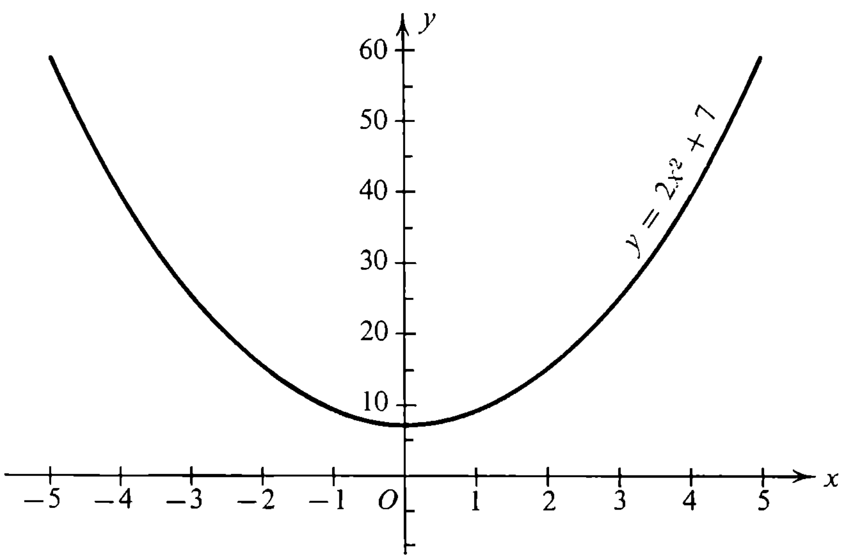


Fig. A

We may now ask whether there exists a *greatest value* among the values which y [determined by the equation (*)] assumes. It is easy to see that such a greatest value of y *does not exist*. Suppose the independent variable x takes on the values

$$x_1 = 1, \quad x_2 = 10, \quad x_3 = 100, \quad x_4 = 1000, \quad \ldots;$$

then the corresponding values of the dependent variable y are

$$y_1 = 9, \quad y_2 = 207, \quad y_3 = 20{,}007, \quad y_4 = 2{,}000{,}007,$$

From this it is evident that there is no greatest y value.

We receive a completely different answer if we ask ourselves whether among the values of y there exists a *smallest.* As the equation (*) shows, y is expressed as the sum of the two terms $2x^2$ and 7. The second term, 7, is a constant number and does not depend on the value of x. The first term, $2x^2$, clearly never becomes negative,[1] this is, *less than zero,* for any value of x. This first term, $2x^2$, can become exactly zero, however, and does so for $x = 0$. Consequently, the first term, $2x^2$, and with it the whole sum, $2x^2 + 7$, takes its least value for $x_0 = 0$. This least or *minimum value* is equal to 7. This is written

$$y_{\min} = 7.$$

Obviously, if there is a lowest point on the graph of a function, then that point represents the smallest value that y can have. The highest point on a graph (if there is a highest point) represents the greatest value that y can have. The fact that the function determined by equation (*) has no greatest y value could be guessed by imagining an extension of our graph to greater values of x.

By such considerations it is easy to show that each of the functions determined by

$$y = 5x^2 + 3, \qquad y = 9x^2 + 4, \qquad y = 2x^2 - 5, \qquad y = 3x^2 - 11$$

possesses similar properties.[2] None has a greatest y value, but each does have a least value, which for all four functions occurs when $x_0 = 0$. These values are, respectively,

$$y_{\min} = 3, \qquad y_{\min} = 4, \qquad y_{\min} = -5, \qquad y_{\min} = -11.$$

2. QUADRATIC TRINOMIALS

The examples just observed are very simple because y is expressed as the sum of two terms, one of which is a *constant,* and the other a *square* (with some fixed positive coefficient), which cannot be negative.

[1] We are considering only real numbers.

[2] While reading through this booklet, the reader will find it helpful to draw graphs of the functions in the examples and problems.

Things become more complicated in the example

$$y = 2x^2 - 12x + 93,$$

where the right-hand side of the equation is a *quadratic trinomial.* In order to be able to employ the same argument as above, we rewrite y in the form

$$y = 2(x^2 - 6x) + 93.$$

Now we complete the square of the expression inside the parentheses, obtaining

$$y = 2(x^2 - 6x + 9) + 93 - 18$$

or

$$y = 2(x - 3)^2 + 75.$$

Now we can employ the same arguments as above, for y is now expressed as the sum of two terms, one of which, namely, 75, is completely independent of x, while the other, $2(x - 3)^2$, certainly can never become negative, although at the point $x = 3$ it is equal to zero. Hence, y has a least value $y_{\min} = 75$, which it reaches at $x = 3$.

On the other hand, a greatest value of y does not exist. We are easily convinced of this if we take, for example,

$$x_1 = 13, \qquad x_2 = 103, \qquad x_3 = 1003, \qquad \ldots.$$

The corresponding values of y are

$$y_1 = 275, \qquad y_2 = 20{,}075, \qquad y_3 = 2{,}000{,}075, \qquad \ldots.$$

We solve the example

$$y = 3x^2 + 24x + 50$$

analogously. Omitting the incidental remarks, we see that

$$\begin{aligned} y &= 3(x^2 + 8x) + 50 \\ &= 3(x^2 + 8x + 16) + 50 - 48 \\ &= 3(x + 4)^2 + 2. \end{aligned}$$

Consequently, y takes its least value at $x_0 = -4$ and for this minimum has the value

$$y_{\min} = 2.$$

Here is one more example:

$$y = 5x^2 - 50x + 39.$$

Here

$$x_0 = 5, \qquad y_{\min} = -86.$$

(We shall always denote by x_0 the value of the independent variable to which the least value of the dependent variable corresponds.)

Of course, a graph for each of the foregoing examples would represent our algebraic considerations visually.

3. MAXIMUM VALUES

It would be fallacious to assume that every quadratic trinomial possesses a least, but not a greatest value. For example,

$$y = -3x^2 + 8$$

has a greatest or *maximum value,*

$$y_{\max} = 8,$$

which it assumes at $x_0 = 0$ (Fig. B). On the other hand, it has no minimum value.

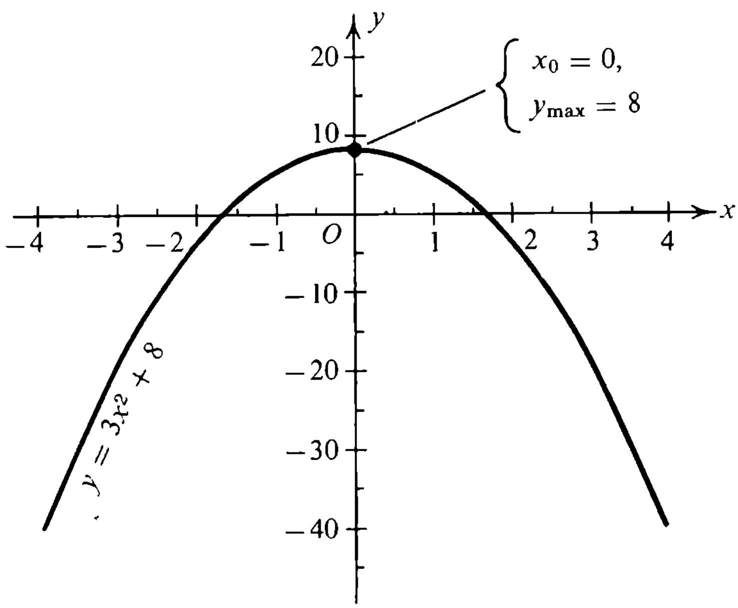


Fig. B

Similarly,

$$y = -4x^2 + 40x - 73$$

has no minimum, but has a maximum value. We can convince ourselves that this is so with the help of the following transformations:

$$\begin{aligned} y &= -4(x^2 - 10x) - 73 \\ &= -4(x^2 - 10x + 25) - 73 + 100 \\ &= -4(x - 5)^2 + 27. \end{aligned}$$

Hence, at $x_0 = 5$,

$$y_{\max} = 27.$$

4. THE FUNDAMENTAL THEOREM

Thus, some quadratic trinomials have a least but no greatest value, while others have a greatest but no least value. The alert reader has probably already noticed that the behavior of the quadratic trinomial is determined by the *sign* of the coefficient of the second-degree term. In order to prove this conclusively, we consider the problem in its general form.

Let the quadratic trinomial

$$y = ax^2 + bx + c$$

be given. Here the coefficients can be any real numbers—positive or negative, as well as zero. The coefficient a of the second-degree term must, however, be different from zero, since otherwise y would contain no term in x^2 and would, consequently, not be a quadratic trinomial.

We transform y in the following manner:

$$y = a\left(x^2 + 2 \cdot \frac{b}{2a} x\right) + c,$$

$$y = a\left(x^2 + 2 \cdot \frac{b}{2a} x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}.$$

For brevity, let us set

$$c - \frac{b^2}{4a} = M,$$

so that we finally get

$$y = a\left(x + \frac{b}{2a}\right)^2 + M.$$

It is important to observe here that M is a fixed number, which is determined by the coefficients a, b, and c and is completely independent of the value of the independent variable x.

We distinguish two cases:

1. If $a > 0$, the first term $a\left(x + \frac{b}{2a}\right)^2$ will never become negative but vanishes at

$$x_0 = -\frac{b}{2a}.$$

Hence, y has a minimum value, M:

$$y_{\min} = M.$$

It has, however, no maximum value.

2. If $a < 0$, by the same considerations we find that

$$y_{\max} = M,$$

and y reaches this value at the point

$$x_0 = -\frac{b}{2a},$$

but there exists no $y_{\min}$.

Least and greatest values are called *extreme values* or *extrema.* Therefore, everything which has been said to this point can be summarized in the following fundamental theorem:

THEOREM. *The quadratic trinomial*

$$y = ax^2 + bx + c$$

has an extreme value, which it assumes at

$$x_0 = -\frac{b}{2a}.$$

This value is a minimum if $a > 0$ and a maximum if $a < 0$. If $y_{\min}$ exists, then $y_{\max}$ does not exist, and inversely.

We notice further that this extreme value, as we saw earlier, is always

$$y_{\text{ext}} = M,$$

or, written in full,

$$y_{\text{ext}} = c - \frac{b^2}{4a}.$$

It is *not necessary,* however, to remember this last equality, because, of course, this number is the value of our trinomial at

$$x = x_0 = -\frac{b}{2a}.$$

Consequently, it is sufficient to substitute into the trinomial the number

$$x_0 = -\frac{b}{2a}$$

in order to get the quantity y_{ext}.

EXAMPLES:

$$\begin{array}{lll} y = 3x^2 - 12x + 8, & x_0 = 2, & y_{\min} = -4; \\ y = -2x^2 + 8x - 3, & x_0 = 2, & y_{\max} = 5; \\ y = 2x^2 + 20x + 17, & x_0 = -5, & y_{\min} = -33. \end{array}$$

The interested student will find it very instructive to construct several graphs and note how changing the sign and value of the constants a, b, c, one at a time, affects the shape and position of the graph of the quadratic trinomial.

2. Applications

5. APPLICATIONS OF THE FUNDAMENTAL THEOREM

We shall now show that the theorem which was proved in section 4 makes possible the solution of a great number of highly diverse problems.

PROBLEM 1. We wish to divide a given positive number A into the sum of two terms such that their product will be maximal. Find the terms.

SOLUTION. We designate by x one of the terms sought. The second term is then equal to $A - x$, and their product is

$$y = x(A - x), \qquad \text{or} \qquad y = -x^2 + Ax.$$

In this way, the problem is reduced to finding the value of x for which this quadratic expression assumes its maximum value. According to the theorem of section 4, such a value exists (since the coefficient of the second degree term is -1, that is, it is negative) and

$$x_0 = \frac{A}{2}, \qquad A - x_0 = \frac{A}{2};$$

that is, the two terms must be equal to one another.

Remark. Let us look at one example. The number 30 admits, among others, the decompositions

$$\begin{aligned}
30 &= 5 + 25, & 5 \cdot 25 &= 125;\\
30 &= 7 + 23, & 7 \cdot 23 &= 161;\\
30 &= 13 + 17, & 13 \cdot 17 &= 221;\\
30 &= 20 + 10, & 20 \cdot 10 &= 200;\\
30 &= 29 + 1, & 29 \cdot 1 &= 29;\\
30 &= 30 + 0, & 30 \cdot 0 &= 0.
\end{aligned}$$

All of these products are less than $15 \cdot 15 = 225$. The reader will find it instructive to plot the points (0, 0), (1, 29), (2, 56), . . . , (5, 125), . . . , (15,225), . . . , (25,125), . . . , (30, 0) and observe the graph.

PROBLEM 2. A wire of length l is given. It is to be bent so that it produces a rectangle which encloses the greatest possible area. Find the dimensions of the rectangle.

SOLUTION. We denote one side of the rectangle by x (Fig. 1). Then the other side is clearly $\frac{l}{2} - x$, and the area

$$S = x\left(\frac{l}{2} - x\right),$$

or

$$S = -x^2 + \frac{l}{2}x.$$

This has its maximum at

$$x_0 = \frac{l}{4},$$

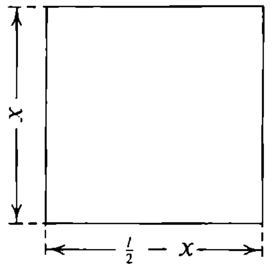


Fig. 1

which is the value of one of the sides of the desired rectangle. The other side is

$$\frac{l}{2} - x_0 = \frac{l}{4};$$

that is, our rectangle turns out to be a *square*.

The conclusion of Problem 2 can be summarized in the following theorem:

THEOREM. *The square possesses the greatest area of all rectangles of the same perimeter.*

Remark. Problem 2 can also be solved easily with the help of the result reached in solving Problem 1. The area which interests us is

$$S = x\left(\frac{l}{2} - x\right).$$

In other words, S is the product of the two factors x and $\frac{l}{2} - x$. But the sum of these factors is

$$x + \left(\frac{l}{2} - x\right) = \frac{l}{2},$$

that is, a number which is not dependent upon the choice of x. Consequently, we are led to the decomposition of the number $\frac{l}{2}$ into the sum of two terms whose product is maximal. But as we know, this product is greatest when both terms are equal, that is, when $x = \frac{l}{4}$.

PROBLEM 3. A fence 200 meters long can be constructed from available material. It is necessary to enclose a rectangular yard of greatest area within this fence, using a factory wall as one side of the yard. Find the dimensions of the rectangular yard.

SOLUTION. We denote the length of one side of the yard perpendicular to the factory wall by x (Fig. 2). The opposite side is

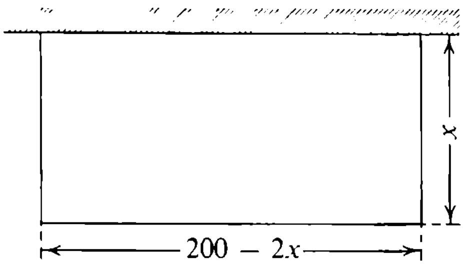


Fig. 2

also x and the third side is then $200 - 2x$; the area is

$$S = x(200 - 2x) \quad \text{or} \quad S = -2x^2 + 200x.$$

According to the theorem of section 4, this attains its maximum value at

$$x_0 = 50.$$

The side of the yard perpendicular to the factory wall must consequently be 50 m. long, while the side parallel to the wall has a length of 100 m.; that is, the yard will be twice as long as it is wide.

Remark. If we had wished to use the solution of Problem 1 in Problem 3, we would not have succeeded immediately, since

$$S = x(200 - 2x)$$

is the product of two factors whose sum equals $200 - x$, and thus is *dependent* on x. In other words, the conditions of Problem 1 are not fulfilled. However, we can use the method of Problem 1 after performing a small trick. Let us consider $z = 2S$ instead of S:

$$z = 2x(200 - 2x);$$

this is the product of two factors whose sum no longer depends on x and whose maximum is attained when

$$2x = 200 - 2x,$$

that is, when $x = 50$. Notice that the functions S and $z = 2S$ assume their maximum at the same value of x.

PROBLEM 4. Let the square $ABCD$ be given (Fig. 3). Let the equal segments Aa, Bb, Cc, Dd be laid off from its vertices and the points a, b, c, d be joined by straight lines. For what value of Aa does the square $abcd$ have the smallest area?

SOLUTION. If we set $AB = l$ and $Aa = x$, then

$$aB = l - x;$$

therefore, according to the Pythagorean theorem

$$\begin{aligned}\overline{ab}^2 &= x^2 + (l - x)^2 \\ &= 2x^2 - 2lx + l^2.\end{aligned}$$

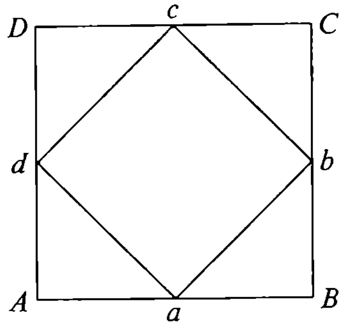


Fig. 3

The area S of the square $abcd$ is just equal to $\overline{ab}^2$, however. Hence

$$S = 2x^2 - 2lx + l^2,$$

and we get the minimum value for S at

$$x_0 = \frac{l}{2}.$$

Thus, the points a, b, c, and d must be located at the mid-points of the sides of the square $ABCD$.

PROBLEM 5. A steamer and a yacht leave points A and B in the indicated directions (Fig. 4) at the same time. Their velocities are $v_S = 40$ km./hr. and $v_Y = 16$ km./hr., respectively. After what period of time is the distance between them least, if $AB = 145$ km.?

SOLUTION. Let the positions t hours after departure of the steamer and the yacht be S and Y, respectively. Then

$$AS = 40t \text{ km.}, \qquad BY = 16t \text{ km.};$$

therefore, according to the Pythagorean theorem,

$$\begin{aligned}SY &= \sqrt{BS^2 + BY^2} \\ &= \sqrt{(145 - 40t)^2 + (16t)^2},\end{aligned}$$

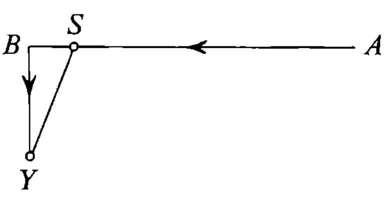


Fig. 4

from which it follows that

$$SY = \sqrt{1856t^2 - 11{,}600t + 21{,}025}.$$

This square root has its minimum at the same value of t for which the expression underneath the radical sign,

$$z = 1856t^2 - 11{,}600t + 21{,}025,$$

is least, that is, at $$t = \frac{11{,}600}{3{,}712} = 3\frac{1}{8} \text{ hr.}$$

Thus, the steamer and the yacht are separated from one another by the shortest distance 3 hours, 7 minutes, and 30 seconds after their departure from A and B.

PROBLEM 6. A rectangle of greatest area is to be inscribed in a given circle. Find the dimensions of the rectangle.

SOLUTION. We denote the radius of the circle by R and the side AB of the rectangle we seek by x (Fig. 5). According to the Pythagorean theorem,

$$BC = \sqrt{4R^2 - x^2},$$

from which we get the expression

$$S = x\sqrt{4R^2 - x^2}$$

for the area S which interests us. This takes its maximum at the same x as $y = S^2$ does. If we set $x^2 = z$ in

$$y = x^2(4R^2 - x^2),$$

we get
$$\begin{aligned} y &= z(4R^2 - z) \\ &= -z^2 + 4R^2z. \end{aligned}$$

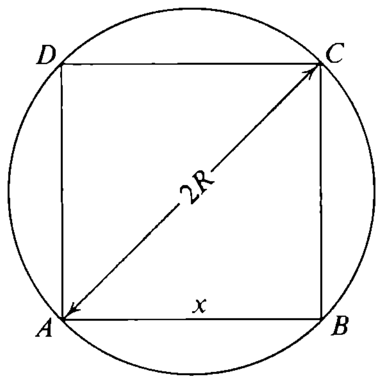


Fig. 5

Hence, $y_{\max}$ is reached when $z = 2R^2$, that is, when $x = R\sqrt{2}$.

We could also have found this value of x without introducing the quantity z if we had been guided by the fact that y is the product of two factors with the constant sum $4R^2$. Thus, according to the result of Problem 1, it follows that $x^2 = 2R^2$ and $x = R\sqrt{2}$.

Noticing that for $AB = x = R\sqrt{2}$

$$BC = R\sqrt{2},$$

we see that the rectangle sought must be a square. Consequently, we have proved the following theorem:

THEOREM. *Of all rectangles which can be inscribed in the same circle, the square has the greatest area.*

PROBLEM 7. A cylinder with greatest lateral area is to be inscribed in a given sphere. Find the dimensions of the cylinder.

SOLUTION. We denote the radius of the sphere by R and the radius and altitude of the desired cylinder by r and h, respectively (Fig. 6). The lateral area of the cylinder is then

$$S = 2\pi rh.$$

As is seen from Figure 6, lengths R, r, and $\frac{1}{2}h$ are connected by the relation

$$\frac{1}{4}h^2 + r^2 = R^2.$$

From this it follows that

$$h = 2\sqrt{R^2 - r^2},$$

and, consequently, that

$$S = 4\pi r\sqrt{R^2 - r^2}.$$

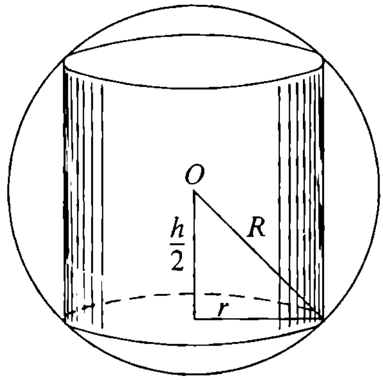


Fig. 6

If we set $y = S^2$ as in Problem 6, we obtain

$$y = 16\pi^2[r^2(R^2 - r^2)].$$

If we now introduce a new independent variable $x = r^2$, then

$$y = 16\pi^2[x(R^2 - x)],$$

and $y_{\max}$ (as well as $S_{\max}$) is reached at $x_0 = \frac{R^2}{2}$, that is, at

$$r = \frac{R}{\sqrt{2}} = R\frac{\sqrt{2}}{2}$$

Knowing r, we can easily find $h = R\sqrt{2}$. Noticing further that $h = 2r$ in the cylinder sought, we see that any cross section which contains the axis of this cylinder must be a square.

PROBLEM 8. A cylinder with greatest lateral area is to be inscribed in a given right circular cone. Find the dimensions of the cylinder.

SOLUTION. We designate the radius of the base and the altitude of the cone by R and H, respectively, and the radius and altitude

of the desired cylinder by r and h, respectively. Thus, $AB = R$, $OA = H$, $AA_1 = r$, and $O_1A_1 = h$ (Fig. 7). The lateral area of the cylinder is then

$$S = 2\pi rh.$$

From the similarity of the triangles OAB and O_1A_1B we derive the proportion

$$\frac{h}{H} = \frac{R-r}{R},$$

from which it follows that

$$h = \frac{H}{R}(R-r)$$

and

$$S = 2\pi\frac{H}{R}[r(R-r)].$$

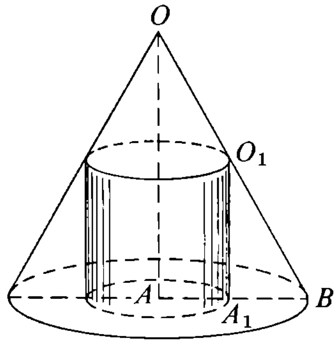


Fig. 7

This function reaches its maximum at $r_0 = \frac{1}{2}R$. Hence, the altitude of the desired cylinder is

$$h_0 = \frac{H}{R}(R - r_0) = \frac{1}{2}H.$$

PROBLEM 9. In the triangle ABC (Fig. 8), where should a straight line ab be constructed parallel to the base AB, so that the area of the rectangle $abcd$ will be maximal?

SOLUTION. We set

$$AB = L, \qquad ab = l, \qquad bc = h$$

and let H be the altitude CD to the side AB of the triangle ABC. From the similarity of the triangles ABC and abC we derive the proportion

$$\frac{l}{L} = \frac{H-h}{H},$$

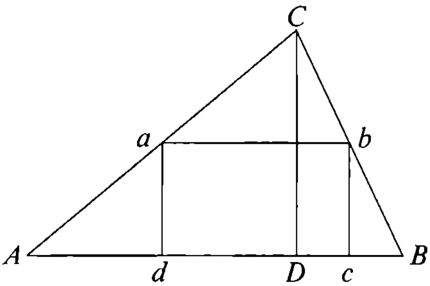


Fig. 8

from which it follows that

$$l = \frac{L}{H}(H-h).$$

Since the area of the rectangle $abcd$, which interests us, is $S = hl$, it follows that

$$S = \frac{L}{H}[h(H-h)].$$

Hence, $S_{\max}$ is reached at $h_0 = \frac{1}{2}H$.

3. Further Theorems and Applications

7. THEOREMS DERIVED FROM PROBLEM 1

Let us return again to Problem 1, considered in section 5. Its solution leads to the following theorem:

THEOREM 1. *The geometric mean of two positive numbers is not greater than their arithmetic mean:*

$$\sqrt{xy} \leq \frac{x+y}{2}. \tag{*}$$

Proof. Let x and y be two positive numbers whose sum is A. The numbers $\frac{1}{2}A$ and $\frac{1}{2}A$ have this sum. Since these two numbers are equal, their product is greater than the product of every other pair of numbers with the same sum (as was proved in Problem 1); in particular, for the pair of numbers x and y,

$$xy \leq \left(\frac{A}{2}\right)^2;$$

the equality sign must be present because $x = y = \frac{1}{2}A$ was not excluded. Recalling that $A = x + y$, we see that

$$xy \leq \left(\frac{x+y}{2}\right)^2,$$

which is equivalent to the inequality (*). The proof shows that *the equality sign holds if and only if $x = y$.*

This theorem can be proved, however, in yet another way, without reference to the results of Problem 1. Namely, we can write the inequality (*) in the equivalent form

$$0 \leq x - 2\sqrt{xy} + y,$$

and in this form it is obvious, since

$$x - 2\sqrt{xy} + y = (\sqrt{x} - \sqrt{y})^2 \geq 0.$$

This proof also establishes that in (*) the equality holds if and only if $x = y$.

PROBLEM 10. A given positive number P is to be divided into two positive factors, so that their sum is minimal. Find the factors.

SOLUTION. Let P be represented in any way as the product of two positive factors x and y,

$$P = xy \qquad (x > 0, y > 0).$$

Then by the inequality (*)

$$x + y \geq 2\sqrt{P}.$$

Thus, with no choice of factors can their sum be *less* than $2\sqrt{P}$. But choosing them equal, that is, letting $x = \sqrt{P}, y = \sqrt{P}$, we get $2\sqrt{P}$ as the sum. Hence, $2\sqrt{P}$ is the desired minimum, which is obtained if and only if[1] the two factors are equal to one another.

If we observe that $y = \frac{P}{x}$, then we see that our result can be stated in the form of the following theorem:

THEOREM 2. *The function determined by*

$$z = x + \frac{P}{x} \qquad (P > 0)$$

(*in which the independent variable x can take only positive values*) *reaches its minimum, $z_{\min}$, at $x_0 = \sqrt{P}$ and only at this value of x.*

PROBLEM 11. A picture AB hangs on a wall above eye level. At what distance from the wall must an observer stand in order that the angle θ subtended by the picture will be the greatest?

SOLUTION. Let K be the point of intersection of the wall with the horizontal line which runs through the eye O of the observer (Fig. 9). The distance sought is then OK. Call this distance x, and let

$$KA = a,$$

$$KB = b.$$

If we denote the angles KOA and KOB by α and β, respectively, then

$$\theta = \beta - \alpha.$$

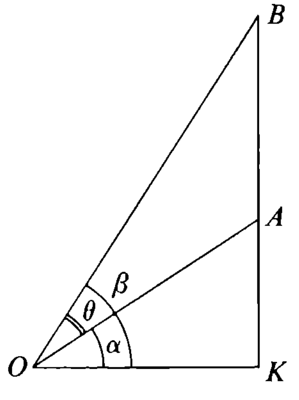


Fig. 9

[1] If x and y are not equal, then, by the last sentence on page 40, we would have the strict inequality $x + y > 2\sqrt{P}$.

From this it follows that[1]

$$\tan \theta = \tan (\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \alpha \tan \beta}.$$

But

$$\tan \alpha = \frac{a}{x}, \qquad \tan \beta = \frac{b}{x}.$$

Consequently,

$$\tan \theta = \frac{b - a}{x + \dfrac{ab}{x}}.$$

Since the maximum of the angle θ is reached when its tangent is maximal, our problem comes to the determination of a value of x for which the fraction

$$\frac{b - a}{x + \dfrac{ab}{x}}$$

is greatest. Since the numerator is constant, we must make the denominator

$$x + \frac{ab}{x}$$

as small as possible. According to Theorem 2, the value of x which is sought is

$$x_0 = \sqrt{ab}.$$

8. GENERALIZATION OF THEOREM 1 OF SECTION 7

Theorem 1 of section 7 permits an important generalization which is of great theoretical interest. This generalization, which will enable us to solve many more problems on maxima and minima, can be formulated as follows:

THEOREM. *The geometric mean of any number of positive numbers is not greater than their arithmetic mean; that is,*

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}. \qquad (*)$$

The equality sign holds if and only if all the numbers $x_1, x_2, \ldots, x_n$ *are equal:* $x_1 = x_2 \ldots = x_n$.

[1] For a discussion of this formula, see any trigonometry book.

Proof. We shall give an ingenious proof, although not a simple one, which represents an unusual variation of mathematical induction. (The proof is due to the distinguished French mathematician A. L. Cauchy, 1789–1857.)

If $n = 2$, our theorem corresponds to the theorem already proved in section 7.

Let $n = 4$. Then according to the theorem already proved,

$$\sqrt[4]{x_1x_2x_3x_4} = \sqrt{\sqrt{x_1x_2} \cdot \sqrt{x_3x_4}} \leq \sqrt{\frac{x_1 + x_2}{2} \cdot \frac{x_3 + x_4}{2}}$$

$$\leq \frac{\dfrac{x_1 + x_2}{2} + \dfrac{x_3 + x_4}{2}}{2};$$

that is,

$$\sqrt[4]{x_1x_2x_3x_4} \leq \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

Hence, inequality (*) is proved for $n = 4$.

Now let $n = 8$. According to what has already been proved,

$$\sqrt[8]{x_1x_2 \cdots x_8} = \sqrt{\sqrt[4]{x_1x_2x_3x_4} \cdot \sqrt[4]{x_5x_6x_7x_8}}$$

$$\leq \sqrt{\frac{x_1 + x_2 + x_3 + x_4}{4} \cdot \frac{x_5 + x_6 + x_7 + x_8}{4}},$$

and hence

$$\sqrt[8]{x_1x_2 \cdots x_8} \leq \frac{\dfrac{x_1 + x_2 + x_3 + x_4}{4} + \dfrac{x_5 + x_6 + x_7 + x_8}{4}}{2},$$

which proves inequality (*) for $n = 8$.

Analogously, we can prove this theorem for $n = 16$, $n = 32$, $n = 64$, and in general, for $n = 2^m$ by induction on m.[1]

We shall give this induction in detail. Suppose that for the natural number m we have the inequality

$$\sqrt[2^m]{x_1x_2 \cdots x_{2^m}} \leq \frac{x_1 + x_2 + \cdots + x_{2^m}}{2^m}, \qquad (**)$$

[1] For a discussion of the method of mathematical induction, see the booklet by I. S. Sominskii in this series.

and let us consider a system of 2^{m+1} positive numbers $x_1, x_2, \ldots, x_{2^{m+1}}$.

It is clear that

$$\sqrt[2^{m+1}]{x_1 \cdots x_{2^{m+1}}} = \sqrt{\sqrt[2^m]{x_1 \cdots x_{2^m}} \cdot \sqrt[2^m]{x_{2^m+1} \cdots x_{2^{m+1}}}}$$

Using inequality (**) and the analogous inequality

$$\sqrt[2^m]{x_{2^m+1} \cdots x_{2^{m+1}}} \leq \frac{x_{2^m+1} + \cdots + x_{2^{m+1}}}{2^m}$$

we obtain

$$\sqrt[2^{m+1}]{x_1 \cdots x_{2^{m+1}}} \leq \sqrt{\frac{x_1 + \cdots + x_{2^m}}{2^m} \cdot \frac{x_{2^m+1} + \cdots + x_{2^{m+1}}}{2^m}},$$

and by the use of Theorem 1 of section 7 we find that

$$\sqrt{\frac{x_1 + \cdots + x_{2^m}}{2^m} \cdot \frac{x_{2^m+1} + \cdots + x_{2^{m+1}}}{2^m}} \leq \frac{\dfrac{x_1 + \cdots + x_{2^m}}{2^m} + \dfrac{x_{2^m+1} + \cdots + x_{2^{m+1}}}{2^m}}{2};$$

from this it follows that

$$\sqrt[2^{m+1}]{x_1 \cdots x_{2^{m+1}}} \leq \frac{x_1 + \cdots + x_{2^{m+1}}}{2^{m+1}}.$$

Thus, inequality (**) is proved for all natural numbers m.

Let us now take n to be a number not of the form 2^m. Then let us choose m so great that

$$2^m > n.$$

If we set

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = A$$

and add $2^m - n$ numbers $\quad x_{n+1} = A, \quad x_{n+2} = A, \quad \ldots, \quad x_{2^m} = A$

to our numbers $x_1, x_2, \ldots, x_n$, then, by what has been proved,

$$\sqrt[2^m]{x_1 \cdots x_n x_{n+1} \cdots x_{2^m}} \leq \frac{x_1 + \cdots + x_n + x_{n+1} + \cdots + x_{2^m}}{2^m},$$

from which it follows that

$$\sqrt[2^m]{x_1 \cdots x_n A^{2^m - n}} \leq \frac{x_1 + \cdots + x_n + (2^m - n)A}{2^m}.$$

However, since

$$x_1 + x_2 + \cdots + x_n = nA,$$

we have

$$\frac{x_1 + \cdots + x_n + (2^m - n)A}{2^m} = \frac{nA + (2^m - n)A}{2^m} = A.$$

Hence, the preceding inequality takes the form

$$\sqrt[2^m]{x_1 x_2 \cdots x_n A^{2^m - n}} \leq A,$$

from which by raising to the power 2^m we get

$$x_1 x_2 \cdots x_n A^{2^m - n} \leq A^{2^m}$$

and, consequently,

$$x_1 x_2 \cdots x_n \leq A^n.$$

Therefore,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq A = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In order to complete the proof, it remains to be verified that in inequality (*) the equality sign holds if and only if

$$x_1 = x_2 = \cdots = x_n.$$

The first of these assertions is obviously true, because if $x_1 = x_2 = \cdots = x_n = a$, then both parts of inequality (*) are equal to a. It is harder to prove the second assertion: if the equality sign holds in inequality (*), then

$$x_1 = x_2 = \cdots = x_n.$$

We shall prove this by assuming the contrary and showing that this assumption leads to a contradiction. Thus, let

$$\sqrt[n]{x_1x_2\cdots x_n} = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

with the numbers $x_1, x_2, \ldots, x_n$ not all equal. Since we may order these numbers any way we wish, we may assume that

$$x_1 \neq x_2.$$

Let us now consider the set of numbers

$$y_1, y_2, y_3, \ldots, x_n,$$

in which all numbers from the third one on are the same as those in the original set, $x_1, x_2, x_3, \ldots, x_n$, but

$$y_1 = y_2 = \frac{x_1 + x_2}{2}.$$

It is clear that $y_1 + y_2 = x_1 + x_2$, so that

$$\frac{y_1 + y_2 + x_3 + \cdots + x_n}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

It follows that

$$\sqrt[n]{x_1x_2\cdots x_n} = \frac{y_1 + y_2 + x_3 + \cdots + x_n}{n}.$$

But from the inequality (*) already proved

$$\sqrt[n]{y_1y_2x_3\cdots x_n} \leq \frac{y_1 + y_2 + x_3 + \cdots + x_n}{n},$$

and this, together with the preceding equality, gives

$$\sqrt[n]{y_1y_2x_3\cdots x_n} \leq \sqrt[n]{x_1x_2x_3\cdots x_n},$$

which, in turn, gives

$$y_1y_2 \leq x_1x_2.$$

But this last relation contradicts Theorem 1 of section 7, for by that theorem the product of *equal* factors, y_1y_2, must be *strictly greater* than the product of *unequal* factors, x_1x_2, which have the same sum $(y_1 + y_2 = x_1 + x_2)$.

The theorem is now completely proved.

9. ARITHMETICAL APPLICATIONS

The theorem proved in section 8 gives the possibility of solving two problems which will be considered in this section.

PROBLEM 12. A given positive number is to be decomposed into n positive terms so that their product is maximal. Find the terms.

SOLUTION. Let the numbers $x_1, x_2, \ldots, x_n$ be positive numbers such that

$$x_1 + x_2 + \cdots + x_n = A.$$

Then by the theorem of section 8,

$$x_1 x_2 \cdots x_n \leq \left(\frac{A}{n}\right)^n.$$

The product of the arbitrarily chosen terms can, therefore, in no case be greater than $\left(\frac{A}{n}\right)^n$. On the other hand, for $x_1 = x_2 = \cdots = x_n = \frac{A}{n}$ we clearly obtain a product that equals $\left(\frac{A}{n}\right)^n$. Consequently, the desired terms must be equal to one another.

By the theorem of section 8, this is the *only* possible solution.

PROBLEM 13. A given positive number P is to be factored into n positive factors whose sum is minimal. Find the factors.

SOLUTION. The solution is analogous to that of Problem 12. If

$$x_1 x_2 \cdots x_n = P,$$

then, according to the theorem of section 8,

$$x_1 + x_2 + \cdots + x_n \geq n \sqrt[n]{P}.$$

Hence the sum of any chosen factors cannot be less than $n\sqrt[n]{P}$. Now, since

$$x_1 = x_2 = \cdots = x_n = \sqrt[n]{P}$$

gives us a sum which is equal to $n\sqrt[n]{P}$, the desired factors must all equal one another.

Here, also, the solution is unique.

10. GEOMETRICAL APPLICATIONS

The results of section 9 permit the solution of an entire series of concrete problems. We shall cite several examples.

In the problems below, the solution we find will in each case be the only possible one. This follows readily from what has been said above, and we shall not refer to it again.

PROBLEM 14. A cylinder of greatest volume is to be inscribed in a given sphere. Find the dimensions of the cylinder.

SOLUTION. Using the notation of Problem 7, we can express the volume of the required cylinder in the form

$$V = \pi r^2 h.$$

As we have seen in Problem 7, $h = 2\sqrt{R^2 - r^2}$. Hence,

$$V = 2\pi r^2 \sqrt{R^2 - r^2}.$$

If we set $z = \frac{1}{4\pi^2} V^2$, we get

$$z = r^4 (R^2 - r^2).$$

Note that z takes its maximum for the same r as does V. Since

$$\frac{1}{4} z = \frac{r^2}{2} \cdot \frac{r^2}{2} \cdot (R^2 - r^2),$$

$\frac{1}{4} z$ is the product of three factors whose sum is equal to R^2. Hence, z will then take its greatest value when the three factors are equal, that is, when r fulfills the condition

$$\frac{r^2}{2} = \frac{r^2}{2} = R^2 - r^2,$$

whence

$$r = R\sqrt{\frac{2}{3}}.$$

PROBLEM 15. A cylinder of greatest volume is to be inscribed in a given right circular cone. Find the dimensions of the cylinder.

SOLUTION. Using the notation of Problem 8, we can express the volume of the required cylinder in the form

$$V = \pi r^2 h.$$

In Problem 8 we found $h = \frac{H}{R}(R - r)$. Consequently,

$$V = \pi \frac{H}{R} r^2 (R - r).$$

V reaches its maximum at the same r as the function

$$z = \frac{r}{2} \cdot \frac{r}{2} \cdot (R - r),$$

which is the product of three factors with a constant sum, R. Hence, $z_{\max}$ occurs when

$$\frac{r}{2} = \frac{r}{2} = R - r,$$

that is, for $r = \frac{2}{3} R$.

PROBLEM 16. A right circular cone of greatest volume is to be inscribed in a given sphere. Find the dimensions of the cone.

SOLUTION. Let the radius of the sphere be R and the radius of the base and the altitude of the cone be r and h, respectively. It is clear from Figure 10 that $r = AB$ is the mean proportional between the segments BD and BC. Since $BD = h$ and $BC = 2R - h$, it follows that

$$r^2 = h(2R - h).$$

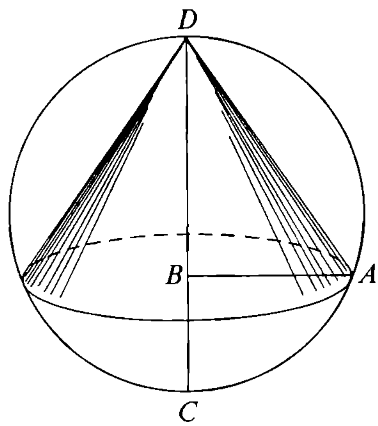


Fig. 10

Since the volume sought is expressed by

$$V = \frac{1}{3} \pi r^2 h,$$

we have $V = \frac{\pi}{3} h^2 (2R - h)$. Let us replace V by the function

$$z = \frac{h}{2} \cdot \frac{h}{2} \cdot (2R - h).$$

This is the product of three factors with a constant sum, $2R$, and, therefore, attains its maximum value where

$$\frac{h}{2} = \frac{h}{2} = 2R - h, \text{ so that } h = \frac{4}{3} R.$$

Since the values of h for which z and V assume their maximum values are the same, our solution is

$$h = \frac{4}{3} R; \qquad r = \frac{2}{3} \sqrt{2} R.$$

PROBLEM 17. A lamp hangs from a pulley over the center of a round table. At what height must this lamp be fixed in order for the illumination of the table edge to be greatest?

SOLUTION. We introduce the notation of Figure 11. It is known from physics that the intensity of light I at the point A is given by the formula

$$I = k\frac{\sin\varphi}{l^2},$$

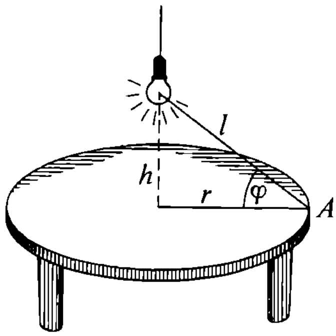


Fig. 11

where k is some constant factor of proportionality. Since

$$\cos\varphi = \frac{r}{l},$$

we get

$$I = \frac{k}{r^2}\sin\varphi\cdot\cos^2\varphi.$$

In place of I we consider the quantity

$$z = \frac{r^4}{k^2}I^2.$$

It is clear that $z_{\max}$ and $I_{\max}$ are attained simultaneously. Now

$$z = \sin^2\varphi\cdot\cos^4\varphi,$$

and from this it follows that

$$\frac{1}{4}z = (1-\cos^2\varphi)\cdot\frac{\cos^2\varphi}{2}\cdot\frac{\cos^2\varphi}{2}.$$

The maximum of z is reached when

$$1-\cos^2\varphi = \frac{\cos^2\varphi}{2} = \frac{\cos^2\varphi}{2},$$

that is, when

$$\cos\varphi = \sqrt{\frac{2}{3}}.$$

For this φ

$$\tan\varphi = \frac{\sqrt{2}}{2},$$

and since

$$h = r\tan\varphi,$$

the height sought is

$$h_0 = r\frac{\sqrt{2}}{2} \approx 0.7r.$$

PROBLEM 18. We are given a rectangular tin sheet measuring 80 cm. by 50 cm. Equal squares are to be cut from the corners, so that the open box formed by bending up the edges contains the greatest possible volume.

SOLUTION. We denote the sides of the squares which are cut out by x (Fig. 12). It is not difficult to see that the volume V of the box will be given by

$$V = x(80 - 2x)(50 - 2x).$$

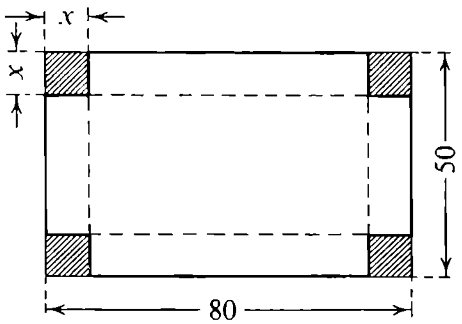


Fig. 12

Here it is not possible to ascertain the desired maximum $V_{\max}$ by the method which we have been using, namely, setting

$$z = 4x(80 - 2x)(50 - 2x)$$

and then setting the factors equal. This is doomed to failure here, since the equation

$$80 - 2x = 50 - 2x$$

is unsolvable.

For this reason, we introduce in the last factor a constant multiplier k, whose numerical value we shall determine later. Thus, instead of V, we consider the function

$$kV = x(80 - 2x)(50k - 2kx).$$

The sum of the factors on the right-hand side is still not a constant; therefore, we multiply the first factor by $2k + 2$ and consider the function

$$z = [(2k + 2)x][80 - 2x][50k - 2kx].$$

Here the sum of the factors is constant for any choice of k and equals $80 + 50k$. Hence, the function z (and with it V) takes on its maximum value when

$$(2k + 2)x = 80 - 2x = 50k - 2kx.$$

Thus, for the determination of x we get two equations:

$$(2k + 2)x = 80 - 2x,$$
$$(2k + 2)x = 50k - 2kx.$$

These equations have the following solutions:

$$x = \frac{40}{k+2}, \qquad x = \frac{25k}{2k+1}.$$

If the problem is to be solvable, these values of x must coincide; that is, we must have

$$\frac{40}{k+2} = \frac{25k}{2k+1}. \tag{*}$$

The fact that we can choose the number k freely now proves to be of help. That is, we choose k so that the condition (*) is fulfilled. Thus, we view the relation (*) as an equation to determine k. If we solve this equation, we find two values for k, namely,

$$k_1 = 2, \qquad k_2 = -\frac{4}{5}.$$

If our formulas are to be meaningful in terms of the problem, the factor

$$50 - 2x,$$

which goes into the expression for V, must be positive. Also, if the method of "equating the factors" is to be applicable to the determination of $z_{\max}$, all the factors must be positive. In particular, the factor

$$50k - 2kx = k\,(50 - 2x)$$

must be positive. The expressions

$$k(50 - 2x) \qquad \text{and} \qquad (50 - 2x)$$

are simultaneously positive, however, only if k is positive.

Hence, we must choose the value

$$k = 2.$$

With this value of k, we find

$$x = 10.$$

This is the solution of the problem.

11. SUMMARY

As we have seen, each and every problem which leads to the determination of an extremal value of the quadratic trinomial

$$y = ax^2 + bx + c$$

can be solved using elementary algebraic means.

In those cases in which the problem leads to the determination of maximum and minimum values of a more complicated function, we succeed in reaching the full solution only with the help of some artificial device, which must be found specifically for each problem.

It is natural to ask ourselves if general methods exist for the determination of the extremal values of functions of any kind, not just quadratic trinomials. Such methods do exist, but, as was said in the Introduction, their study requires a knowledge of calculus.

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G. E. SHILOV, a professor at Moscow State University, is internationally known for his research in functional analysis. I. P. NATANSON is a professor at Leningrad University. He has written well-known books in mathematical analysis and has published a large number of research papers.